Further results:

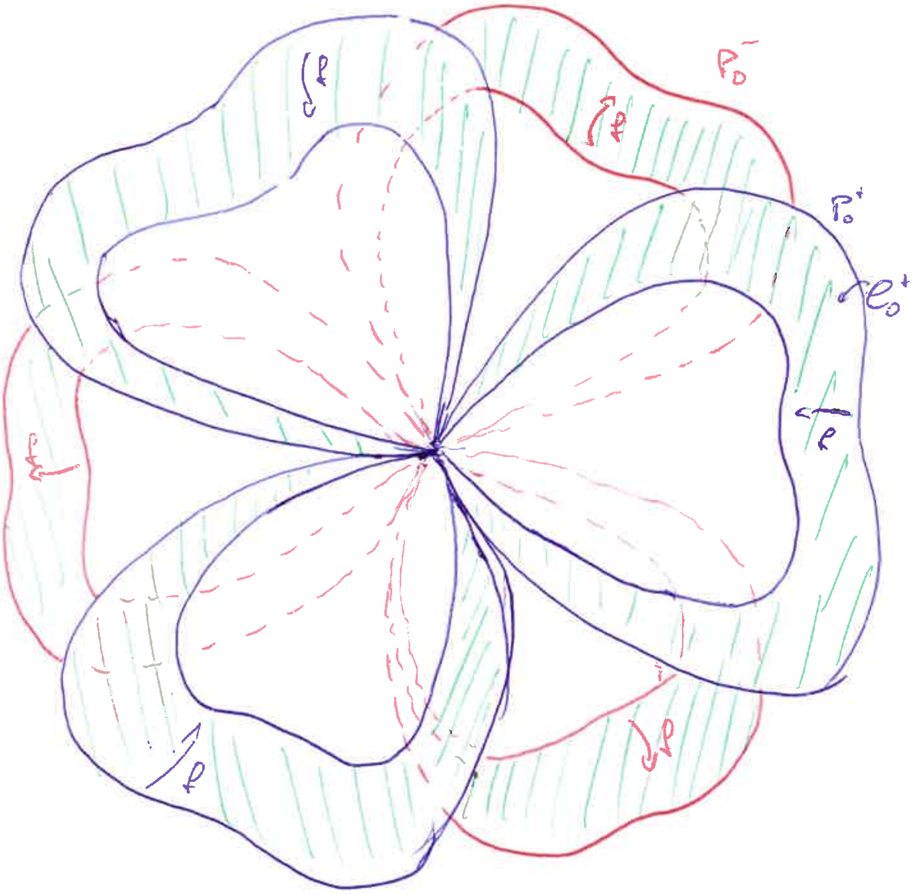
Topological classification (Camacho)

Theorem: Let $f \in (\mathbb{C},0)$ be a tangent to the identity germ with multiplicity $r+1$. Then f is topologically conjugated to $z \mapsto z - z^{r+1}$.

(or equivalently to the time-1-flow of $\frac{dz}{dt} = z^{r+1} \rightsquigarrow f_1(z) = z/\sqrt[r]{1-rz^r}$)

Idea of the proof.

The idea is to construct a fundamental domain for f and g two maps with multiplicity 1 and same multiplicity, and construct an homeo between such domains that respects the dynamics on the boundary



P_j^+: attracting petals.

P_j^-: repelling petals.

$e_j^+ = \overline{P_j^+} \setminus f(P_j^+)$

$e_j^- = \overline{f(P_j^-)} \setminus P_j$.

$$D = \bigcup_j e_j^+ \cup \bigcup_j (e_j^- \setminus K)$$
fund. domain

$$K = \bigcup_{j=1}^{r} P_j^+$$

Rem: The conjugacy is also quasi-conformal.

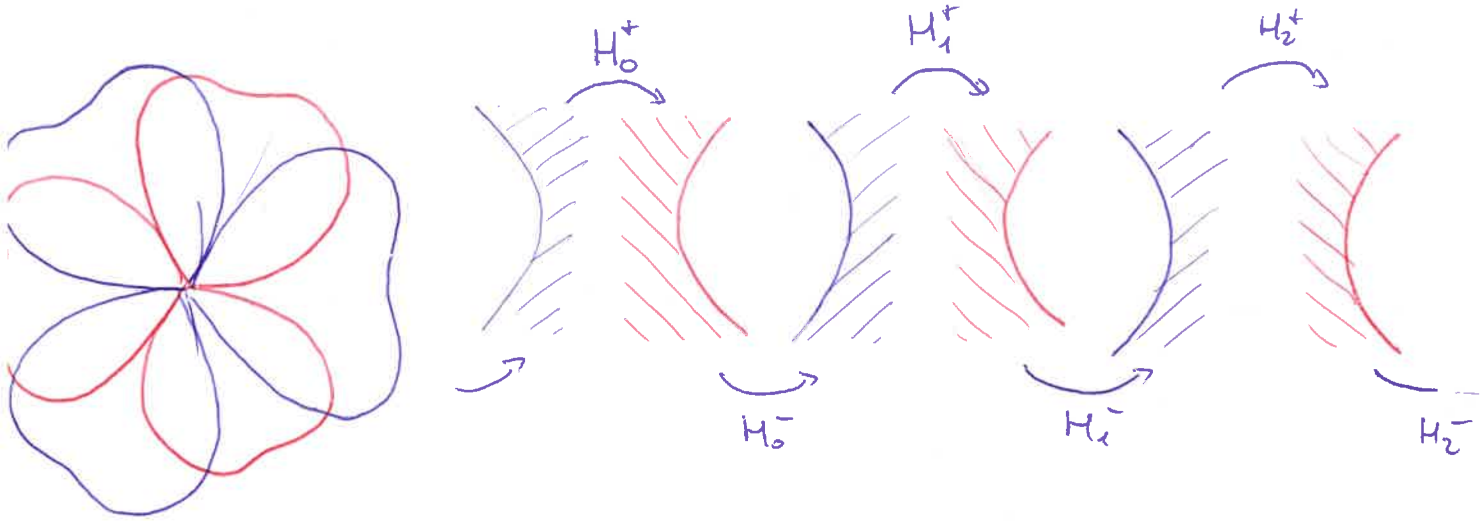
See [Buff - Hubbard, Dynamics in One Complex Variable]

Analytic classification (Écalle/Voronin).

The analytic classification is much more complicated, even to state:

Recall we constructed Petals $P_0^+, P_0^-, P_1^+, P_1^-, \ldots, P_{2-1}^+, P_{2-1}^-$, together with Fatou coordinates $\phi_j^+ : P_j^+ \xrightarrow{\simeq} V_j^+$; $\phi_j^- : P_j^- \xrightarrow{\simeq} V_j^-$, where:

$\phi_j^\pm \circ f(z) = \phi_j^\pm(z)+1$, $V_j^+ \supset \{\mathrm{Re}\, w >> 0\}$, $V_j^- \supset \{\mathrm{Re}\, w << 0\}$.



Consider the maps $H_j^+ : \phi_j^+(P_j^+ \cap P_j^-) \to \phi_j^-(P_j^+ \cap P_j^-)$ $\quad H_j^+ = \phi_j^- \circ (\phi_j^+)^{-1}$

$H_j^- : \phi_j^-(P_j^- \cap P_{j+1}^+) \to \phi_{j+1}^+(P_j^- \cap P_{j+1}^+)$, $H_j^- = \phi_{j+1}^+ \circ (\phi_j^-)^{-1}$.

The maps H_j^+ (resp H_j^-) can be extended to holomorphic maps on domains of the form $\{\mathrm{Im}\, w >> 0\}$ (resp. $\{\mathrm{Im}\, w << 0\}$). (Lifted horn maps)

Moreover, we have that $H_j^\pm(w+1) = H_j^\pm(w) + 1$

Hence the maps H_j induce through the projection $pr(w) = e^{2\pi i w}$, maps $h_j^\pm$, called horn maps, defined on punctured neighborhoods of 0 and ∞ in $\hat{\mathbb{C}}$ respectively.

The maps extend holomorphically at 0 (resp ∞), and define holomorphic invertible germs (convergent)

Rem: Fatou coordinates are uniquely determined up to post-compositions with a translation: $\tilde{\phi}_j^{\pm} = \tau_j^{\pm} \circ \phi_j^{\pm}$ $\tau_j^{\pm}(w) = w + b_j^{\pm}$

- Hence lifted horn maps are uniquely defined up to pre/post composition with translations. $\tilde{H}_j^+ = \tau_j^- \circ H_j^+ \circ (\tau_j^+)^{-1}$ $\tilde{H}_j^- = \tau_{j+1}^+ \circ H_j^- \circ (\tau_j^-)^{-1}$
- And horn maps are uniquely defined up to pre/post multiplication by constants. $\tilde{h}_j^+ = \mu_j^- \circ h_j^+ \circ (\mu_j^+)^{-1}$ $\tilde{h}_j^- = \mu_{j+1}^+ \circ h_j^- \circ (\mu_j^-)^{-1}$. $(*)$

where $\mu_j^{\pm}(z) = e^{2\pi i b_j^{\pm}} \cdot z$.

Def: The Ecalle-Voronin invariant of f is given by the collection $(h_0^+, h_0^-, \ldots, h_{r-1}^+, h_{r-1}^-)$ of horn maps, up to the equivalence relation described by $(*)$ above.

Denote by $\lambda_j^{\pm}$ the multiplier of $h_j^{\pm}$ at $0 / \infty$.

Fact: $\prod_{k=0}^{r-1} \lambda_k^+ \lambda_k^- = e^{4\pi \,\mathrm{Resit}(f)}$, where $\mathrm{Resit}(f) = \frac{r+1}{2} - \frac{\beta}{\alpha^2}$.

well defined for e Ecalle-Voronin invariant.

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$\mathrm{Resit}(f^n) = \frac{1}{n} \mathrm{Resit}(f)$

formal invariant (index)

Theorem (Ecalle-Voronin). $f, g : (\mathbb{C}, 0) \circlearrowleft$ tangent to the identity germs. f and g are analytically conjugated if and only if they have the same multiplicity $r+1$ & the same index β/α^2) and the same Ecalle-Voronin invariant.

Reference: Buff-Hubbard.

Parabolic fixed points: the general case:

$f(z) = e^{2\pi i \frac{p}{q}} z(1+o(1))$.

Prop: $f:(\mathbb{C},0)\circlearrowleft$ so that $f'(0)=e^{2\pi i\frac{p}{q}}=:\lambda$. Then f is holomorphic (formally, topologically) conjugated to $z\mapsto \lambda z$ if and only if $f^q\equiv \mathrm{id}$.

Proof: $\Rightarrow$ let ϕ so that $\phi\circ f\circ\phi^{-1}(z)=\lambda z$. Then

$\phi\circ f^{q}\circ\phi^{-1}(z) = (\phi\circ f\circ\phi^{-1})^{q}(z) = \lambda^q z = \mathrm{id}(z)$.

$\Leftarrow$ Suppose $f^q\equiv\mathrm{id}$. Set $\phi(z)=\frac{1}{q}\cdot\sum_{j=0}^{q-1}\frac{f^j(z)}{\lambda^j}$.

Then $\phi\circ f(z) = \frac{1}{q}\cdot\sum_{j=0}^{q-1}\frac{f^{j+1}(z)}{\lambda^j} = \lambda\cdot\phi(z)$ and $\phi'(0)=\frac{1}{q}\cdot\sum\frac{\lambda^j}{\lambda^j}=1$.

Hence f is conjugate to $z\mapsto\lambda z$. □

Now, let f as above, not conjugated to $z\mapsto\lambda z$ (i.e. non linearizable, i.e., not finite order, i.e. parabolic).

In this case $f^q(z)=z(1+a_r z^r+o(z^r))$.

Prop. f as above, then $q\mid r$.

Proof. Let v be an attracting direction for f^q. Pick any orbit $z_0, z_q, z_{2q},\dots$ converging to 0 tangent to v.

Then $z_1=f(z_0), z_{q+1}=f(z_q),\dots$ converges to 0 tangent to λv.

Hence multiplication by λ permutes the attracting directions, and $q\mid r$.

One can deduce formal, topological and analytical classifications in these cases from the analogous statements for tangent to the identity case.

Example: $f(z) = -z + az^2 + bz^3$.

$$f^2(z) = z - az^2 - bz^3 + a\left(-z + az^2 + bz^3\right)^2 + b\left(-z + az^2 + bz^3\right)^3$$

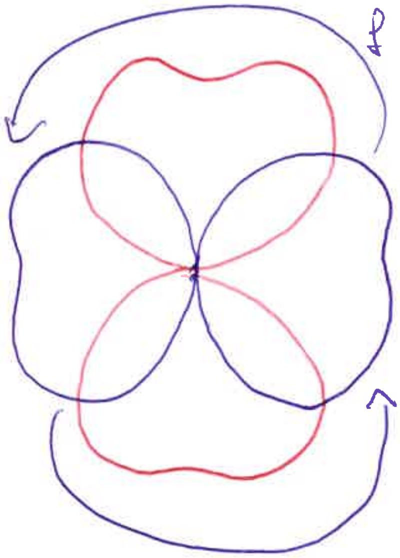
$$= z - az^2 + az^2 - bz^3 - 2a^2z^3 - bz^3 + (a^3 - 2ab)z^4 + 3abz^4 + 2a^2bz^5 + (-3a^2b + 3b^2)z^5 + o(z^5)$$

$$= z - 2(b + a^2)z^3 + a(b + a^2)z^4 + b(3b - a^2)z^5 + o(z^5).$$

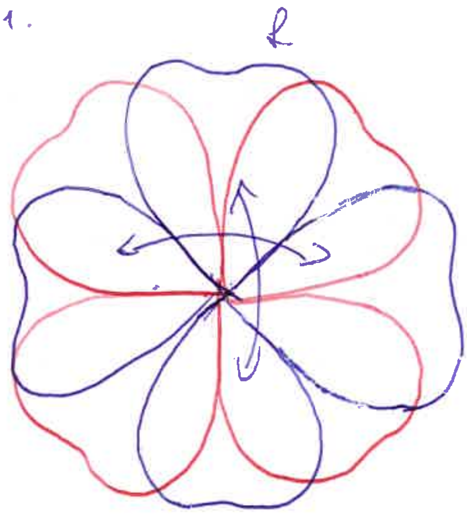
Hence if we denote by $r+1$ the multiplicity of f^2, we have:

$r = 2$ if $b + a^2 \neq 0$, $\quad r = 4$ if $-b = a^2$ $(\neq 0)$.

$a = -1$, $b \neq -1$.



$a = b = -1$.



Prop: $f, g : (\mathbb{C}, 0) \circlearrowleft$ with the same multiplier $\lambda = e^{2i\pi \frac{p}{q}}$. Then $f \simeq g \iff f^q \simeq g^q$.

($\simeq$ analytic, formal or topological conjugacy)

Proof: $\Rightarrow$ obvious.

$\Leftarrow$. Assume $\phi \circ f^q = g^q \circ \phi \rightsquigarrow g^q = \phi \circ f^q \circ \phi^{-1} = (\phi \circ f \circ \phi^{-1})^q$.

Up to replacing f by $\phi \circ f \circ \phi^{-1}$, we may assume $f^q = g^q$. Set

$\psi = \frac{1}{q} \sum_{k=0}^{q-1} g^{q-k} \circ f^k$. Then $\psi \circ f = g \circ \psi$, and $\psi'(0) = 1$. $\square$

Corollary: Formal classification: $f(z) = \lambda z(1 - z^{qs} + \beta z^{2qs})$

Topological classification: $f(z) = \lambda z(1 - z^{qs})$

Analytic classification: in terms of Écalle-Voronin invariants ...